

# Renormalization group approach to the spin-1 Bose gas

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**Abstract.** A field theoretical renormalization group approach at two loop level is applied to the homogeneous spin-1 Bose gas in order to investigate the order of the phase transition. The beta function of the system with  $d = 4 - \epsilon$  dimensions is determined up to the third power of the coupling constants and the system's free energy on the border of the classical stability is given in next to leading order. It is found that the phase transition of the interacting spin-1 Bose gases with weak spin-dependent coupling constant values is of first order.

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## 1 Introduction

Bose–Einstein condensation (BEC) of dilute, interacting, scalar atomic gases is widely believed to be a continuous phase transition. Mean-field theory results based on perturbation theory are however contradicting. The simplest approximations call for a continuous phase transition [1], while according to the more sophisticated Hartree–Fock (Popov) approximation BEC is of first order [2–4]. The contradicting results are understood to be the consequence of critical phenomena. Namely, close to the critical point, the different Feynman diagrams develop to the same magnitude, and a perturbative treatment becomes inaccurate. The different renormalization group calculations are indicating that BEC is a continuous phase transition falling into the same universality class as the O(2) model of field theory [5–8].

In experiments made with dilute gases of alkali atoms in optical traps [9–14] the particles have internal spin degrees of freedom. Such systems at low temperatures can be modeled by Hamiltonians with multiple coupling constant interactions [15–17] in the s-wave scattering limit. For such a situation, with multiple coupling constants, (and in the homogeneous case) the order of the transition is not necessarily of continuous type, as e.g. in the case of the field theoretical  $\phi^4$  model with cubic anisotropy, where fluctuations can induce the transition to be of first order [18, 19].

In this paper we treat the problem of the homogeneous, spin-1 Bose gas, where two coupling constants arise naturally in the low energy limit. In the absence of an external magnetic field depending on the magnitude of these two parameters two possible Bose–Einstein condensed phases

exist, namely, the ferromagnetic phase, in which the system favors a macroscopic magnetization and the polar phase, in which the system prefers no magnetization. The gas condensing to the ferromagnetic phase is called the ferromagnetic gas, while the other is the polar gas (see Eqs. (4) and the discussion below). We assume zero external magnetic field. For such a system mean-field theory results are also contradicting. In the Hartree–Fock approximation [20] the phase transition of both the polar and ferromagnetic gases is of first order. The jump in parameters is the function of the bigger coupling constant, which is responsible for non spin-flip scatterings. However this strongly first order type of transition is considered to be an artifact of the Hartree–Fock approximation such as in the case of the scalar Bose gas. On the other hand the Hartree approximation [21], which is in a way a simpler mean field approximation yields a continuous Bose condensation in the polar case, while a first order one in the case of a ferromagnetic Bose gas. In the latter case the jump is a function of the smaller coupling constant, responsible for spin flip scatterings. (The ratio of the two coupling constants is typically in the order of  $10^{-2}$ .) It is important to note that the Hartree approximation is supplying a continuous BEC in the case of scalar Bose gases [1], and it is related to the leading order approximation of the  $1/N$  expansion of the O( $N$ ) symmetric model in field theory [22, 23]. Because of the above ambiguities a renormalization group approach is worked out to study the order of the phase transition of the homogeneous, spin-1 Bose gas. The formulation is based on the assumption that the phase transition is continuous. In this case the universal quantities (to leading order in the coupling constants) and the infrared (IR) behavior of the system can be calculated from a classical field theory obtained by restricting

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the quantum fields to the zero Matsubara frequency sector [8,24]. The main conclusion of the paper is that this assumption leads to contradiction which indirectly proves that the transition is of first order.

The paper is organized as follows. In Section 2 the model of the homogeneous spin-1 Bose gas is mapped to the corresponding classical field theory by neglecting the nonzero Matsubara frequency components of the quantum fields. In Section 3 the renormalization program with minimal subtraction and dimensional regularization is worked out up to the order of two loops for the classical field theory. In Section 4 the beta function of the model is given up to the third power of the coupling constants and the renormalized free energy is derived up to one loop level. The critical properties of the system are discussed also in this section. Some summary is left to Section 5.

## 2 Classical field theory

The effective Hamiltonian of the low temperature, homogeneous, spin-1 Bose gas can be written as

$$\mathcal{H} = \int d^3x \left[ \frac{\hbar^2}{2M} \nabla \Psi_r^\dagger(x) \nabla \Psi_r(x) - \mu \Psi_r^\dagger(x) \Psi_r(x) \right] + \frac{1}{2} \int d^3x d^3x' V_{r's'}^{rs}(x-x') \Psi_r^\dagger(x) \Psi_{r'}^\dagger(x') \Psi_{s'}(x') \Psi_s(x), \quad (1)$$

with  $M$  the mass of an atom and  $\mu$  the chemical potential. The bosonic field operator  $\Psi_r^\dagger(x)$  creates an atom at position  $x$  with spin projection  $r \in \{+1, 0, -1\}$ , and the operator  $\Psi_r(x)$  destroys it. Automatic summation over repeated spin indices is implicitly assumed throughout the paper. The two-particle interaction is modeled by s-wave scattering, with the interaction potential [15,16]

$$V_{r's'}^{rs}(x-x') = \delta^{(3)}(x-x') \frac{4\pi\hbar^2}{M} \left[ a_0 (\mathcal{P}_0)_{r's'}^{rs} + a_2 (\mathcal{P}_2)_{r's'}^{rs} \right], \quad (2)$$

where  $a_0$  and  $a_2$  are the s-wave scattering lengths in the total hyperfine channel  $F=0$  and  $F=2$ , respectively. The matrices  $\mathcal{P}_0$  and  $\mathcal{P}_2$  are the projection operators in the 9-dimensional tensor product space of the spin variables projecting to the subspaces of total hyperfine spin 0 and 2, respectively. The matrix  $\mathcal{P}_1$ , projecting to the total hyperfine spin-1 subspace is omitted from equation (2), since it is antisymmetric in its indices and therefore does not appear in the Hamiltonian (1). The projection matrices can be obtained from the following linear equations:

$$\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 = 1, \quad (3a)$$

$$-2\mathcal{P}_0 - \mathcal{P}_1 + \mathcal{P}_2 = \mathbf{S}_1 \cdot \mathbf{S}_2, \quad (3b)$$

$$4\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 = (\mathbf{S}_1 \cdot \mathbf{S}_2)^2, \quad (3c)$$

where  $\mathbf{S}_i$  ( $i=1,2$ ) is the spin operator of the  $i$ th atom. Using the usual spin-1 operators in the basis of  $S_z$  eigenvectors one can get from equations (3) the projection

matrices  $\mathcal{P}_0$ ,  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Only expressing the needed two,  $\mathcal{P}_0$  and  $\mathcal{P}_2$  are given by:

$$(\mathcal{P}_0)_{RS} = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4a)$$

$$(\mathcal{P}_2)_{RS} = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \quad (4b)$$

with  $R=5-3r-r'$  and  $S=5-3s-s'$ .

As shown by Ho [15] and Ohmi and Machida [16], if  $a_2 > a_0$  the low temperature, equilibrium phase has a macroscopic wave function with zero net magnetization, called as the polar phase in analogy to the  $^3\text{He}$  case, while if  $a_0 > a_2$  the low temperature phase has a wave function with macroscopic magnetization, called as the ferromagnetic phase.

In this paper we focus on the determination of the order of the phase transition. Supposing first that the possible transitions to the polar or to the ferromagnetic phases of the spin-1 Bose gas are continuous phase transitions and restricting ourselves to universal quantities of the system, the problem can be mapped to a classical field theory with the following bare action [8,24]:

$$\mathcal{S}[\varphi^*, \varphi] = \int d^d x \left[ \frac{1}{2} \partial_\mu \varphi_r^*(x) \partial_\mu \varphi_r(x) + \frac{1}{2} m^2 \varphi_r^*(x) \varphi_r(x) + \frac{c_{rs,r's'}}{4} \varphi_r^*(x) \varphi_{r'}^*(x) \varphi_{s'}(x) \varphi_s(x) \right], \quad (5)$$

with  $\varphi_r(x)$  the 3-component, complex, classical field at the  $d$ -dimensional position  $x$ , and  $\partial_\mu$  is the  $d$ -dimensional gradient. The dimension of the system is generalized from 3 to  $d$  for later purposes. Further on we set  $\hbar = k_B = 1$ . The bare mass of the field theory is denoted by  $m^2 \equiv -2M\mu$ , and the tensor structure of the interaction term takes the form of

$$c_{rs,r's'} = \frac{16\pi^2 a_0}{\lambda_B^2} (\mathcal{P}_0)_{r's'}^{rs} + \frac{16\pi^2 a_2}{\lambda_B^2} (\mathcal{P}_2)_{r's'}^{rs}, \quad (6)$$

with  $\lambda_B = \sqrt{2\pi/MT}$  being the de-Broglie wavelength.

The parameters  $m^2$  and  $c_{rs,r's'}$  of the bare action (5) are easily obtained from the finite temperature action [2] corresponding to the Hamiltonian (1) by simply replacing the imaginary time dependent fields with time independent ones and by integrating out the imaginary time. This

procedure is clearly a restriction to the zero Matsubara frequency sector of the full field theory. The effects of the higher Matsubara frequency components would provide a physical cutoff to the field theory and change the value of the above parameters. These questions are not discussed in this paper since the order of the phase transition is not sensitive for the actual value of the above quantities.

It is more convenient to introduce the matrices  $X \equiv \mathcal{P}_0 + \mathcal{P}_2$  and  $Y \equiv \mathcal{P}_2 - 2\mathcal{P}_0$  and to express the bare interaction (6) with the help of them:

$$c_{rs,r's'} = c_n X_{rs,r's'} + c_s Y_{rs,r's'}, \quad (7)$$

with

$$c_n = \frac{16\pi^2}{\lambda_B^2} \frac{a_0 + 2a_2}{3}, \quad (8a)$$

$$c_s = \frac{16\pi^2}{\lambda_B^2} \frac{a_2 - a_0}{3}. \quad (8b)$$

Note that in the case  $a_2 > a_0$ , i.e. in the polar case  $c_s > 0$ , while in the ferromagnetic case ( $a_0 > a_2$ )  $c_s < 0$ .

### 3 Renormalization up to order of two loops

In the following we suppose that on the critical surface, all of the renormalized masses are zero. This assumption is valid in the absence of an external magnetic field and when the system is in a paramagnetic and non-magnetized phase above the (spinor) Bose–Einstein condensation. The standard renormalization group program of the massless theory with dimensional regularization and minimal subtraction [22, 23] will be carried out for the classical field theory, described by the bare action (5) and interaction (7). The free propagator corresponding to the quadratic part of the bare action then reads as:

$$G_{(0)rs}(p) = \delta_{rs} G_{(0)}(p) = \delta_{rs} \frac{2}{p^2}, \quad (9)$$

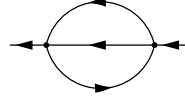
which is just twice the value of the free propagator in a theory with real fields.

The  $n$ -point vertex function is denoted by  $\Gamma^{(n)}(p_i, c)$ , with  $c$  the bare interaction (7),  $p_i$  ( $i \in \{1 \dots n\}$ ) are the wave-numbers of the vertex function.  $\Gamma^{(n)}$  has  $n$  spin indices. The spin indices will be omitted most of the time for notational simplicity. The renormalization conditions for the vertex functions then read as:

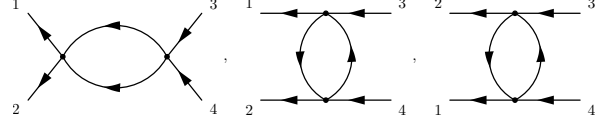
$$\Gamma_R^{(n)}(p_i, \kappa, g) = Z^{n/2}(g) \Gamma^{(n)}(p_i, c), \quad (10a)$$

where  $Z(g)$  is the field renormalization constant,  $\kappa$  is the momentum scale, where the renormalization is made, and  $g$  stands for the dimensionless renormalized coupling constant, having the same index structure as the bare one (7). The connection between the bare and renormalized coupling constants is established by

$$c_{ij,kl} \equiv \kappa^\epsilon g_{0ij,kl} = \kappa^\epsilon G_{ij,kl}(g), \quad (10b)$$



**Fig. 1.** The second order divergent graph contributing to the 2 point function.



**Fig. 2.** The divergent graphs contributing to the 4 point function at 1 loop order.

with  $\epsilon = 4 - d$ ,  $g_0$  is the dimensionless, bare coupling constant and  $G_{ij,kl}(g)$  the coupling constant renormalization function. The renormalization constants are expanded as power series of the renormalized coupling constants  $g_{ij,kl}$ , according to

$$Z(g) = 1 + b^{(2)}(g, g), \quad (11a)$$

$$G_{ij,kl}(g) = g_{ij,kl} + a_{ij,kl}^{(1)}(g, g) + a_{ij,kl}^{(2)}(g, g, g). \quad (11b)$$

Here we have introduced the following symbolic notation for scalar and tensorial quadratic quantities:

$$b(g, g) = b_{i_2 j_2, k_2 l_2}^{i_1 j_1, k_1 l_1} g_{i_1 j_1, k_1 l_1} g_{i_2 j_2, k_2 l_2}, \quad (12a)$$

$$a_{ij,kl}^{(1)}(g, g) = \left( a_{ij,kl}^{(1)} \right)_{i_2 j_2, k_2 l_2}^{i_1 j_1, k_1 l_1} g_{i_1 j_1, k_1 l_1} g_{i_2 j_2, k_2 l_2}, \quad (12b)$$

respectively, and similarly for the cubic  $a_{ij,kl}^{(2)}(g, g, g)$ . The 2-point and 4-point vertex functions are expanded as:

$$\Gamma_{ij}^{(2)}(p, g_0) = p^2 \delta_{ij} \left[ 1 - \Sigma^{(2)}(g_0, g_0) + \mathcal{O}(g_0^3) \right],$$

$$\Gamma_{ij,kl}^{(4)}(p_i, g_0) = \kappa^\epsilon \left[ g_{0ij,kl} + d_{ij,kl}^{(1)}(g_0, g_0) \right] \quad (13a)$$

$$+ d_{ij,kl}^{(2)}(g_0, g_0, g_0) + \mathcal{O}(g_0^4). \quad (13b)$$

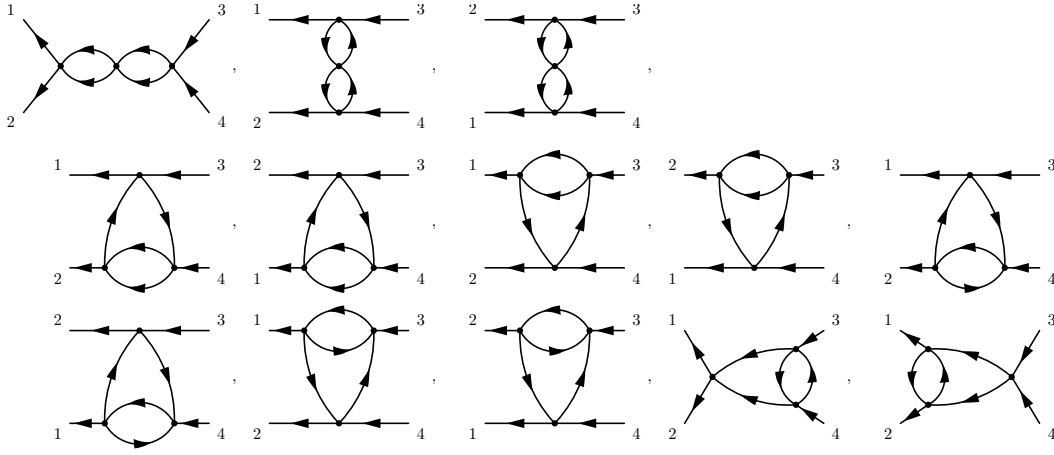
In equation (13a)  $p^2 \Sigma^{(2)}(g, g)$  is the two loop contribution to the self-energy. (The first order term is momentum independent and cancelled, since we assume that the renormalized masses are zero.) The constants  $d^{(1)}$  and  $d^{(2)}$  are the corresponding 1 and 2 loop contributions to the four-point function.

The divergent graph (up to the order of two loops) contributing to the two point function is plotted in Figure 1. With the requirement that the renormalized vertex function (10a) for  $n = 2$  is finite, one obtains, that

$$b^{(2)}(g, g) = -\frac{N_d^2}{\epsilon} \frac{g_{im,kl} g_{lk,mi}}{3}, \quad (14)$$

with  $N_d = 2/(4\pi)^{d/2} \Gamma(d/2)$  the usual factor appearing after each momentum integration.  $\Gamma(s)$  is the Euler gamma function with argument  $s$ .

The divergent graphs contributing to the four point function at 1 loop level are drawn in Figure 2, while at 2 loop order are drawn in Figure 3. With the requirement that the renormalized four point function



**Fig. 3.** The divergent graphs contributing to the 4 pont function at 2 loop order.

[Eq. (10a) with  $n = 4$ ] is finite, the renormalization constants can be obtained. The one loop contribution is

$$a^{(1)}(g, g) = N_d \frac{2}{\epsilon} \left[ g^{(a)} + 2(g^{(b)} + g^{(c)}) \right], \quad (15a)$$

while the two loop contribution reads as:

$$\begin{aligned} a^{(2)}(g, g, g) = & N_d^2 \frac{4}{\epsilon^2} \left\{ g^{(d)} + 2(g^{(g)} + g^{(h)}) + g^{(i)} + g^{(j)} \right. \\ & + g^{(k)} + g^{(l)} + 2 \left( g^{(m)} + g^{(n)} + g^{(o)} + g^{(p)} \right) + 8(g^{(e)} + g^{(f)}) \\ & \left. - \epsilon \left[ -g \frac{g_{nm,op} g_{po,mn}}{6} + g^{(g)} + g^{(h)} + \frac{1}{2}(g^{(i)} + g^{(j)} + g^{(k)} \right. \right. \\ & \left. \left. + g^{(l)} + 2[g^{(m)} + g^{(n)} + g^{(o)} + g^{(p)}]) \right] \right\}, \quad (15b) \end{aligned}$$

with the notations

$$g_{ij,kl}^{(a)} = g_{im,kn} g_{mj,nl}, \quad (16a)$$

$$g_{ij,kl}^{(b)} = g_{ij,mn} g_{nm,kl}, \quad (16b)$$

$$g_{ij,kl}^{(c)} = g_{kj,mn} g_{nm,il}, \quad (16c)$$

$$g_{ij,kl}^{(d)} = g_{im,kn} g_{mo,np} g_{oj,pl}, \quad (16d)$$

$$g_{ij,kl}^{(e)} = g_{ij,mn} g_{nm,op} g_{po,kl}, \quad (16e)$$

$$g_{ij,kl}^{(f)} = g_{kj,mn} g_{nm,op} g_{po,il}, \quad (16f)$$

$$g_{ij,kl}^{(g)} = g_{im,kn} g_{mj,op} g_{po,nl}, \quad (16g)$$

$$g_{ij,kl}^{(h)} = g_{im,op} g_{kn,po} g_{mj,nl}, \quad (16h)$$

$$g_{ij,kl}^{(i)} = g_{ij,nm} g_{ko,mp} g_{on,pl}, \quad (16i)$$

$$g_{ij,kl}^{(j)} = g_{kj,nm} g_{io,mp} g_{on,pl}, \quad (16j)$$

$$g_{ij,kl}^{(k)} = g_{io,mp} g_{oj,pn} g_{kl,nm}, \quad (16k)$$

$$g_{ij,kl}^{(l)} = g_{ko,mp} g_{oj,pn} g_{il,nm}, \quad (16l)$$

$$g_{ij,kl}^{(m)} = g_{ij,nm} g_{kn,po} g_{mp,ol}, \quad (16m)$$

$$g_{ij,kl}^{(n)} = g_{kj,nm} g_{in,po} g_{mp,ol}, \quad (16n)$$

$$g_{ij,kl}^{(o)} = g_{io,pm} g_{oj,np} g_{mn,kl}, \quad (16o)$$

$$g_{ij,kl}^{(p)} = g_{ko,pm} g_{oj,np} g_{mn,il}. \quad (16p)$$

## 4 Critical properties

The critical properties of the massless theory can be studied with the help of the  $\beta$  function, defined as

$$\frac{dG_{ij,kl}}{dg_{mn,op}} \beta_{mn,op} = -\epsilon G_{ij,kl}. \quad (17)$$

The  $\beta$  function can be easily expressed by inverting the matrix  $dG/dg$  perturbatively in  $g$ , e.g. with the help of iteration. The result reads as

$$\begin{aligned} \beta = & -\epsilon g + 2N_d \left[ g^{(a)} + 2(g^{(b)} + g^{(c)}) \right] - 4N_d^2 \left[ 2(g^{(g)} \right. \\ & \left. + g^{(h)}) + g^{(i)} + g^{(j)} + g^{(k)} + g^{(l)} + 2(g^{(m)} + g^{(n)} + g^{(o)} + g^{(p)}) \right. \\ & \left. - g \frac{g_{nm,op} g_{po,mn}}{3} \right] + \mathcal{O}(g^4). \quad (18) \end{aligned}$$

The tensorial  $\beta$  function (18) splits into two functions, according to

$$\beta_{ij,kl} = \beta_n(g_n, g_s) X_{ij,kl} + \beta_s(g_n, g_s) Y_{ij,kl}, \quad (19)$$

with  $X$  and  $Y$  defined above equation (7). The corresponding functions are:

$$\begin{aligned} \beta_n(g_n, g_s) = & -\epsilon g_n + 2N_d (7g_n^2 + 4g_n g_s + 4g_s^2) \\ & - 4N_d^2 (24g_n^3 + 22g_n^2 g_s + 39g_n g_s^2 + 20g_s^3) + \mathcal{O}(g^4), \quad (20a) \end{aligned}$$

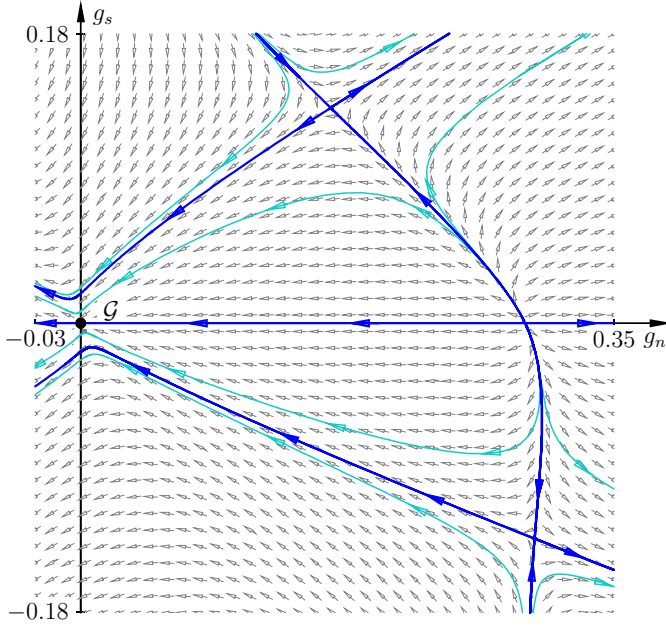
and

$$\begin{aligned} \beta_s(g_n, g_s) = & -\epsilon g_s + 2N_d (6g_n g_s + 3g_s^2) \\ & - 4N_d^2 (28g_n^2 g_s + 28g_n g_s^2 + g_s^3) + \mathcal{O}(g^4). \quad (20b) \end{aligned}$$

These functions are responsible for the flow of the renormalized coupling constants under a change of scale:

$$\rho \frac{dg_{n,s}}{d\rho} = \beta_{n,s}(g_n, g_s) \quad (21)$$

with  $\rho$  being the scale, and  $g_{n,s}(\rho = 1) = g_{n,s}^0$ . The IR behavior ( $\rho \rightarrow 0$ ) of the system can be studied with the help of the IR fixed points of the  $\beta$  function (20).



**Fig. 4.** The flow diagram and fixed point structure at  $d = 4$  ( $\epsilon = 0$ ).

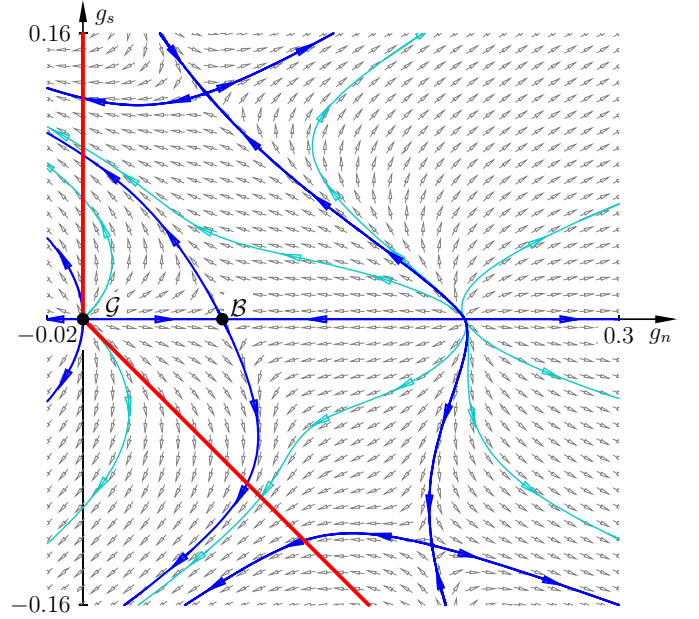
At  $d = 4$  four real fixed points exist. The Gaussian one:  $\mathcal{G}$ , with  $(g_n, g_s) = (0, 0)$ , and three nonphysical fixed points, which are absent at the one loop calculation. The fixed point structure and the coupling constant flow is shown in Figure 4. At  $d = 4 - \epsilon$ , with  $0 < \epsilon \ll 1$  a new real fixed point,  $\mathcal{B}$ , emerges, which is of  $\mathcal{O}(\epsilon)$ . The fixed point up to  $\mathcal{O}(\epsilon^2)$  is  $(g_n, g_s) = (\tilde{g}_n, 0)$ , with

$$\tilde{g}_n = \frac{1}{2N_d} \epsilon \left( \frac{1}{7} + \frac{24}{343} \epsilon + \mathcal{O}(\epsilon^2) \right). \quad (22)$$

This fixed point is the U(3) symmetric fixed point of Bose condensation. It is stable from the direction of the Gaussian one, however it is repulsive through the direction of  $g_s$ . The fixed point structure and the flow diagram is shown in Figure 5. Since there is no attractive fixed point (for  $g_s \neq 0$ ), the trajectories “run away”, which is an indication that both the polar (when  $g_s > 0$ ) and the ferromagnetic (when  $g_s < 0$ ) Bose condensation is of first order. Such fluctuation induced first order transitions are not rare, e.g. the case of a real  $\phi^4$  theory with cubic anisotropy [18,19].

The runaway of the trajectories requires an analysis of the free energy (thermodynamic potential) of the classical system, or as in the terminology of field theory the effective action, which is the generating functional of the vertex functions. The free energy is obtained with the help of the method of steepest descent up to one loop order [22,23]. The zeroth order (tree-graph) contribution is the bare action (5):

$$\frac{1}{V} \Gamma_0[\varphi] = \frac{1}{2} m^2 \varphi_r \varphi_r + \frac{1}{4} c_{rs, r' s'} \varphi_r \varphi_s \varphi_{r'} \varphi_{s'}, \quad (23)$$



**Fig. 5.** The flow diagram and fixed point structure at  $d < 4$  (for  $\epsilon = 0.4$ ). The red lines indicate the boundary of the stability wedge of the zeroth order contribution of the free energy (see below).

with  $\varphi$  being real and homogeneous. The free energy (23) describes a continuous phase transition at  $m^2 = 0$ .  $V = L^d$  is the volume of the system. For  $m^2 < 0$  and  $c_s > 0$  (polar case) the homogeneous field minimizing the potential (23) can be chosen as  $\varphi_r = \varphi \times (0, 1, 0)_r$ , which has zero magnetization, while for  $c_s < 0$  (ferromagnetic case) the minimizing solution can be taken as  $\varphi_r = \varphi \times (1, 0, 0)_r$ , which has maximal magnetization. With these solutions the free energy at tree-graph level takes the following forms:

$$\frac{1}{V} \Gamma_0^{\text{pol}}[\varphi] = \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} c_n \varphi^4, \quad (24a)$$

$$\frac{1}{V} \Gamma_0^{\text{ferr}}[\varphi] = \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} (c_n + c_s) \varphi^4. \quad (24b)$$

In the polar case ( $c_s > 0$ ) the tree graph thermodynamic potential (24a) is bounded from below only if  $c_n > 0$ , while in the ferromagnetic case ( $c_s < 0$ ) equation (24b) is confining only if  $c_n + c_s > 0$ . The line of stability is therefore  $c_n = 0$  for  $c_s > 0$  and  $c_n + c_s = 0$  for  $c_s < 0$ . As can be seen in Figure 5 all interesting trajectories with  $g_s \neq 0$  reach the stability boundaries of the tree graph free energy and go outside the region of stability. It is therefore mandatory to calculate the next order contribution. Following the standard technique [22,25,26] (but with complex fields) the one-loop contribution of the free energy reads as

$$\frac{1}{V} \Gamma_1^{\text{pol}}[\varphi] = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left\{ \log \left[ (k^2 + m^2 + 2c_n \varphi^2)^2 - c_n^2 \varphi^4 \right] + 2 \log \left[ (k^2 + m^2 + (c_n + c_s) \varphi^2)^2 - c_s^2 \varphi^4 \right] \right\} \quad (25a)$$

in the polar case, while

$$\frac{1}{V} \Gamma_1^{\text{ferr}}[\varphi] = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \log \left[ (k^2 + m^2 + 2(c_n + c_s)\varphi^2)^2 - (c_n + c_s)^2\varphi^4 \right] + 2 \log \left[ k^2 + m^2 + (c_n + c_s)\varphi^2 \right] + 2 \log \left[ k^2 + m^2 + (c_n - c_s)\varphi^2 \right] \right\} \quad (25b)$$

in the ferromagnetic case. The integrals appearing in equations (25) are divergent. Carrying out the renormalization scheme renders them finite:

$$\frac{1}{V} \Gamma_1[\varphi] = f(\nu_1) + f(\nu_2) + 2f(\nu_3) + 2f(\nu_4) \quad (26a)$$

both for the polar and ferromagnetic cases, with

$$\nu_1^{\text{pol}} = t + 3g_n\varphi^2, \quad (26b)$$

$$\nu_2^{\text{pol}} = t + g_n\varphi^2, \quad (26c)$$

$$\nu_3^{\text{pol}} = t + (g_n + 2g_s)\varphi^2, \quad (26d)$$

$$\nu_4^{\text{pol}} = t + g_n\varphi^2 \quad (26e)$$

in the polar case and

$$\nu_1^{\text{ferr}} = t + 3(g_n + g_s)\varphi^2, \quad (26f)$$

$$\nu_2^{\text{ferr}} = t + (g_n + g_s)\varphi^2, \quad (26g)$$

$$\nu_3^{\text{ferr}} = t + (g_n + g_s)\varphi^2, \quad (26h)$$

$$\nu_4^{\text{ferr}} = t + (g_n - g_s)\varphi^2 \quad (26i)$$

in the ferromagnetic one. Here we introduced  $t$  the renormalized, dimensionless temperature, and measured the field  $\varphi$  in units of  $\kappa^{1-\epsilon/2}$ , with  $\kappa$  being the scale of the renormalization. The function  $f$  appearing in equation (26a) is given by

$$f(x) = \frac{x^2}{8} \left( \log x + \frac{1}{2} \right). \quad (27)$$

The sum of equations (23) and (26a) gives the free energy of the classical system (5) in next to leading order.

On the border of stability the free energy up to one-loop order (both for the polar and ferromagnetic cases) can be cast to the form (with neglecting terms not depending on  $\varphi$ ):

$$\frac{1}{V} \Gamma[\varphi] = \frac{1}{2} t \varphi^2 + 2f(t + 2|g_s|\varphi^2). \quad (28)$$

The potential (28) describes systems with a first order phase transition for  $0 < |g_s| = \mathcal{O}(\epsilon)$ . It is worth to note that in the Hartree approximation made for the quantum theory of the spin-1 Bose gas directly in 3 dimensions [27,28], or in the equivalent lattice mean-field calculation also made directly in 3 dimensions [29] the phase transition was found to be of first order for small coupling constant values (at least for the ferromagnetic case).

## 5 Summary

In summary, we have studied the critical properties of spin-1 Bose gases with the assumption that the phase transition to the polar and to the ferromagnetic Bose–Einstein condensed phases is of second order. In this case the universal IR behavior of the quantum system (1) can be studied with the help of a classical field theory (5) obtained by restricting the fields to the zero Matsubara frequency sector. The machinery of the field-theoretical renormalization group then was applied to the classical field theory. The  $\beta$  function was determined up to the order of two loops in the massless theory. It was found that the only at least partially stable physical fixed point (for  $d < 4$ ) is the U(3) symmetric one of Bose–Einstein condensation with  $g_s = 0$ . However this fixed point was found to be repulsive towards the direction of  $g_s$ . This indicates that all trajectories of systems with nonzero  $g_s$  tend towards the border of the classical stability wedge with successive scale transformations. The free energy [on the boundary of the stability wedge (28)] of the classical system (5) was determined in next to leading order in the method of steepest descent. It was found that both in the polar and ferromagnetic cases the free energy develops a second local minimum (besides the trivial one), which turns to be a global minimum at a certain temperature above  $t = 0$  (before the second order phase transition sets in). This shows that the classical system described by the bare action (5) exhibits a first order phase transition for  $0 < |g_s| = \mathcal{O}(\epsilon)$ , at least near four dimensions. This contradicts our assumption that the polar and ferromagnetic Bose–Einstein condensations are continuous phase transitions (if they were, the classical description would yield also a continuous phase transition), and shows that the phase transition is indeed of first order both for the polar and ferromagnetic gases and that the jump in the thermodynamic quantities depend on the smaller coupling constant:  $g_s$ . One can regard the presented approach as an indirect proof.

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